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TODA FIELDS ON RIEMANN SURFACES: REMARKS ON THE MIURA TRANSFORMATION

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Abstract. We point out that the Miura transformation is related to a holomorphic foliation in a relative flag manifold over a Riemann Surface. Certain differential operators corresponding to a free field description of W–algebras are thus interpreted as partial connections associated to the foliation.

1. Introduction and background

The Miura transformation plays an important role in the free–field approach to the theory of integrable systems. This is very effective in the case of a cylindrical topology, where it has been shown how the correspondence between solutions to the Toda field equations and free–fields is basically one to one [2], thereby concluding a program initiated by Gervais and Neveu [11]. Free fields are also a powerful tool in describing the relevant W–algebra structures and in addressing quantization problems [8, 5, 11].

However, the above mentioned correspondence becomes problematic as soon as one tries to directly extend it to Riemann Surfaces of higher genus. A "no–go theorem" has been devised in ref. [1] for the uniformization solution to the Liouville equation, and the same analysis applies to the geometric approach to the Toda equations pursued in ref. [3]. These results prompt for an investigation of the meaning of the Miura transformation in presence of a non trivial topology.

Our aim will be to show in what sense the well known local calculations to be recalled below can be recast in a global setting and to exhibit an interpretation for the relations defining the Miura transformation. This should be of interest in connection with the problem of determining to what extent free fields do describe Toda Field theories on a Riemann surface.

1.1. It is well known that classical W-algebras can be described in terms of gauge equivalence classes of chiral connections under lower triangular gauge transformations [7]. The customary W-fields appear when one fixes the gauge to

$$\nabla = \partial + \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & 1 \\ w_n & w_{n-1} & \dots & \dots & 0 \end{pmatrix} dz, \quad T \equiv w_2 \tag{1.1}$$

hereafter called "Drinfel'd-Sokolov" gauge. (We follow [3, 6, 17], a different naming convention was adopted in [1].) This is true in particular for the zero curvature

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representation of the Toda equations in the holomorphic gauge [3, 17]. On the other hand, there are other relevant gauges [6], and in particular the "diagonal gauge"

$$\widetilde{\nabla} = \partial + \begin{pmatrix} p_1 & 1 & 0 & \dots & 0 \\ 0 & p_2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & 1 \\ 0 & \dots & \dots & \dots & p_n \end{pmatrix} dz, \quad p_1 + \dots + p_n = 0 \tag{1.2}$$

gives the above mentioned free fields. It is customary to introduce new variables $a_1, \dots, a_{n-1}$ through

$$\begin{aligned} p_1 &= a_1 \\ p_k &= a_k - a_{k-1}\,, \quad k = 2, \dots, n-1 \\ p_n &= -a_{n-1} \end{aligned}$$

in order to enforce the null trace condition in (1.2). One can connect the two slices (1.1) and (1.2) by a lower triangular gauge transformation

$$\nabla \mapsto \widetilde{\nabla} = \mathbf{N}_-^{-1} \circ \nabla \circ \mathbf{N}_- \tag{1.3}$$

with

$$\mathbf{N}_- = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ a_1 & 1 & 0 & \dots & 0 \\ * & a_2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ * & \dots & * & a_{n-1} & 1 \end{pmatrix} \tag{1.4}$$

(the other terms are determined in terms of $a_1, \dots, a_{n-1}$) which yields a relation

$$w_i = w_i[\{a_j\}, \{\partial a_j\}, \dots, \{\partial^{i-1} a_j\}] \tag{1.5}$$

expressing the W-fields in terms of the free fields. This is precisely the *Miura transformation.* For example, in the rank two case one gets

$$T(z) = -\partial_z a(z) + a(z)^2\,, \tag{1.6}$$

while if $n = 3$

$$\begin{aligned} w_2 &= -\partial_z a_1 + a_1^2 - \partial_z a_2 + a_2^2 - a_1 a_2 \\ w_3 &= \partial_z(\partial_z a_1 - a_1^2) + a_1(\partial_z a_2 - a_2^2) + a_1^2 a_2\,, \end{aligned} \tag{1.7}$$

and so on.

1.2. One is typically interested in a local basis of flat sections

$$\nabla \mathbf{M} = 0\,. \tag{1.8}$$

The preceding gauge transformation is encoded in the (Gauss) factorization

$$\mathbf{M} = \mathbf{N}_-\, \mathbf{B}_+ \tag{1.9}$$

where $\mathbf{N}_-$ is a lower unipotent matrix and $\mathbf{B}_+$ is an upper triangular matrix. This splitting yields a corresponding basis $\mathbf{B}_+$ of flat sections for $\widetilde{\nabla}$ and automatically

provides a solution to (1.5). This is the way the Miura transfomation is hidden in the free field approach to Toda Field theory exploited in [2].

Remark 1.1. The factorization (1.9) implies taking ratios among various entries in the matrix $\mathbf{M}$, thus it is bound to be ill–defined at certain points. This problem is usually dealt with by assuming the Gauss factorization can be performed. We shall characterize the locus where this can actually be done.

1.3. The two $\mathcal{D}$-modules (1.1) and (1.2) can actually be defined in a more general setting. (See, in this connection, ref. [9].) Let C be a compact Riemann surface of genus $g > 1$. It has been shown that (1.1) defines a holomorphic connection on the vector bundle $E = J^{n-1}({K_C}^{-\frac{n-1}{2}})$ of $(n-1)$-jets of holomorphic sections of ${K_C}^{-\frac{n-1}{2}}$, where K_C is the canonical line bundle [3, 16]. On the other hand, (1.2) defines a *meromorphic* connection on the vector bundle $\widetilde{E} = \bigoplus_{r=0}^{n-1} {K_C}^{-\frac{n-1}{2}+r}$ [1, 17]. (Because of Weil's theorem, $\widetilde{E}$ does not admit analytic connections [4, 13].) By the monodromy analysis in [1], which easily carries over to the rank n case, there are no (meromorphic) morphisms from the pair (E, ∇) to $(\widetilde{E}, \widetilde{\nabla})$. This is especially true when the former results from the zero–curvature representation of the Toda equations. Therefore the local procedure leading to the gauge transformation cannot be globally implemented, so that on a Riemann Surface we have two radically distinct objects.

The above observation questions the idea of extending the approach to classical W-algebras via gauge equivalence classes of connections to a higher genus surface. Thus the line we shall stick to in the following will be to consider the vector bundle E and the analytic connection (1.1) as the definition for W-type objects on a Riemann Surface. This is the same one as in ref. [3] and it is in line with other geometric approaches, such as [15]. (Note that this is also the opposite to [17].)

1.4. Let us describe our result in some more detail. (See however the next section for the exact definitions.) We shall show in the following that *the Miura transformation defines a holomorphic foliation on a certain complex manifold X.* More precisely, let $\pi : X \to C$ be the relative complete flag $Fl(E)$, for (E, ∇) the flat holomorphic vector bundle on C introduced before. One can define (relative) Schubert Cells in X in the same fashion as in the standard case. Moreover, there is a natural holomorphic foliation on X induced by the horizontal distribution corresponding to ∇.

Then *the Miura transformation is the solution to the differential system defining this foliation in the affine coordinates on the "Big Cell" $X_0 \subset X$. (X_0 is an affine bundle over C.)* Moreover, the *operator* (1.2) *arises as a partial connection δ associated to the same foliation on X. By construction, the pull-back connection on π^*E obtained from ∇ is adapted to the partial connection δ.*

1.5. The organization of this paper will be as follows. In the next section we set up the notation and describe the various construction we need in the following. Next, we discuss the holomorphic foliation and describe its relation to the Miura transformation. Finally, we briefly discuss some examples and draw our conclusions.

2. Some constructions

2.1. Notation and setup. The compact Riemann surface C of genus $g > 1$ will be kept fixed throughout. Local coordinates on C will customarily be denoted by $z, z_\alpha, z_\beta, \dots$ with domains $U, U_\alpha, U_\beta, \dots$ All bundles and morphisms will be considered in the holomorphic category.

2.1.1. For the sake of simplicity we shall restrict ourselves in the following to the special linear group. We shall use the canonical root decomposition $\mathfrak{g} \equiv \mathfrak{sl}_n(\mathbb{C}) = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ such that the nilpotent subalgebras $\mathfrak{n}^\pm$ correspond to strictly upper (lower) tringular matrices and the Cartan subalgebra $\mathfrak{h}$ to diagonal ones. $\mathfrak{b}^\pm = \mathfrak{h} \oplus \mathfrak{n}^\pm$ are the upper (lower) Borel subalgebras and $N^\pm$, $B^\pm$ are the analytic subgroups corresponding to the subalgebras so far introduced.

2.1.2. We consider the pair (E, ∇) where $E \to C$ is the jet-bundle mentioned before. It is determined by the following condition [3]. Let us temporarily append the subscript "n", so that $E_{(n)} \equiv E$; then E_n is characterized as the extension $(E_1 \equiv \mathcal{O}_C)$

$$0 \longrightarrow K_C^{\frac{1}{2}} \otimes E_{(n-1)} \longrightarrow E_{(n)} \longrightarrow K_C^{-\frac{n-1}{2}} \longrightarrow 0$$

It follows that it is equipped with a complete filtration by subbundles

$$\{0\} \equiv F^0_{(n)} \subset F^1_{(n)} \subset \cdots \subset F^{n-1}_{(n)} \subset F^n_{(n)} \equiv E_{(n)} \tag{2.1}$$

with $F^r_{(n)} \cong K_C^{\frac{n-r}{2}} \otimes E_{(r)}$ and $F^{r+1}/F^r \cong K_C^{\frac{n-1}{2}-r}$, $r = 0, \dots, n-1$, so that $\mathrm{Gr} F^\bullet \cong \widetilde{E}$. Explicit expressions for the cocycle $\{\varphi_{\alpha\beta}\}$ representing E can be found in [3, 16]; what is relevant for us is that they can be put into *lower triangular form.* Thus there are isomorphisms $E|_U \to \mathcal{O}^n_C|_U$ together with bases $e_1, \dots, e_n$ of sections of E on $U \subset C$ such that $F^r|_U$ is identified with the span $\langle e_{n+1-r}, \dots, e_n \rangle$. ∇ is a flat analytic connection whose local expression is given by (1.1) above. According to that, we have $\nabla F^r \subset F^{r+1} \otimes K_C$.

2.1.3. $\pi : Fl(E) \to C$ is the relative flag whose fiber over a point $p \in C$ is the manifold $Fl(E_p)$ of complete flags in E_p. We shall simply set $X \equiv Fl(E)$. On $Fl(E)$ we have the usual universal (tautological) flag $\{0\} \subset S^1 \subset \cdots \subset S^{n-1} \subset \pi^* E$ and a corresponding universal sequence of quotients $\pi^* E \twoheadrightarrow Q^{n-1} \twoheadrightarrow \cdots \twoheadrightarrow Q^1$, where $Q^{n-i} \equiv \pi^* E / S^i$. Recall that if $W^\bullet \in X$ then $S^i_{W^\bullet} = W^i$.

2.1.4. We set $\rho : P \to C$ to be the $\mathrm{SL}_n(\mathbb{C})$ principal bundle corresponding to E. (Since $\det E \cong \mathcal{O}_C$, the structure group is reduced to $\mathrm{SL}_n(\mathbb{C})$.) Furthermore, restricting the action of $\mathrm{SL}_n(\mathbb{C})$ on P to B^+ yields

$$\begin{tikzcd} P \arrow[r, "\tilde{\pi}"] \arrow[d, "\rho"'] & X \arrow[d, "\pi"] \\ C \arrow[r, equal] & C \end{tikzcd}$$

as a relative version of the natural projection $\mathrm{SL}_n(\mathbb{C}) \xrightarrow{B^+} Fl(\mathbb{C}^n)$.

Remark 2.1. Notice that $B^+ \hookrightarrow P \xrightarrow{\tilde{\pi}} X$ *does not* correspond to the reduction given by the filtration (2.1). In fact the latter reduces the structure group to the *lower* Borel B^-.

2.2. The flat connection ∇ determines a horizontal holomorphic integrable distribution $\mathcal{H} \hookrightarrow T_P$. The image $\mathcal{F} \equiv \tilde{\pi}_* \mathcal{H}$ of the horizontal distribution $\mathcal{H}$ is clearly holomorphic and integrable (both by dimensional reasons and because it is the direct image of an integrable distribution), so it defines a holomorphic foliation in X. Moreover, the integrable distribution $\mathcal{H}$ in P satisfies with respect to the fibration $P \xrightarrow{\tilde{\pi}} X$ all the properties of the horizontal distribution of a connection, except that it does not project onto the tangent bundle of the base but only on a foliation thereof, a situation referred to as *partial connection.* [14]

The datum of a partial connection determines an operator on X

$$\delta : \pi^* E \longrightarrow \pi^* E \otimes \mathcal{F}^*$$

satisfying the Leibniz rule in the form

$$\delta(fs) = \tau(df) \otimes s + f\delta(s)$$

where τ is the projection $\tau : \Omega^1_X \to \mathcal{F}^*$ resulting from the injection $\mathcal{F} \hookrightarrow T_X$.

The connection ∇ pulls back to a flat and holomorphic connection on $\pi^* E$ (and $\pi^* P$), denoted by the same symbol. This pull-back connection is *adapted* to the partial connection if $\delta = (1 \otimes \tau) \circ \nabla$. (see ref. [14] and below.)

Remark 2.2. It is interesting to see what the leaves of $\mathcal{F}$ are. Since E is flat, $X \cong \widetilde{C} \times_{\pi_1(C)} Fl(\mathbb{C}^n)$, where $\widetilde{C}$ is the universal cover of C and $\pi_1(C)$ acts on the flag $Fl(\mathbb{C}^n)$ through the holonomy representation. Then the leaves are just the images of $\widetilde{C} \times \{\mathrm{pt}\}$ under the projection $\widetilde{C} \times Fl(\mathbb{C}^n) \to \widetilde{C} \times_{\pi_1(C)} Fl(\mathbb{C}^n)$.

2.3. We set out to characterize geometrically the calculations presented in 1.1, 1.2. Following ref. [10], we use the filtration (2.1) to define a submanifold $X_0 \subset X$. As a set, it is determined by the condition that for any $p \in C$ it consists of those flags $W^\bullet : \{0\} \subset W^1 \subset \cdots \subset W^{n-1} \in X_p = \pi^{-1}(p)$ such that for any i

$$\dim(W^j \cap F_p^{n+i-j}) = i\,. \tag{2.2}$$

It is easy to see that this condition is in fact redundant and the very same locus is characterized by restricting (2.2) to $i = 0$ only. More invariantly, one is led to the equivalent condition that X_0 is the locus where the map

$$\pi^* F^k \hookrightarrow \pi^* E \twoheadrightarrow Q^k$$

has maximal rank [10]. We can condense its geometrical meaning in the following

Lemma 2.3. *X_0 is a relative big cell in X, i.e. for any $p \in C$, $X_0 \cap X_p$ is the big cell in $X_p \cong Fl(\mathbb{C}^n)$, namely the orbit of the lower unipotent subgroup of* $\mathrm{SL}_n(\mathbb{C})$ *through the standard flag. Furthermore, $X_0 \xrightarrow{\pi|_{X_0}} C$ is an affine bundle over C.*

2.3.1. We sketch the proof in order to set some convention needed in the following. Full details can be found in [10]. It follows from 2.1.2 that we can use coordinates in each fiber such that:

(1) The canonical projection $\mathrm{SL}_n(\mathbb{C}) \to Fl(\mathbb{C}^n)$ is realized by sending $g \in \mathrm{SL}_n(\mathbb{C})$ to the flag $\langle g \cdot e_1 \rangle \subset \langle g \cdot e_1, g \cdot e_2 \rangle \subset \cdots \subset \mathbb{C}^n$ where $\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \dots$ is the standard flag, whose stabilizer is the upper Borel B^+.
(2) The filtration corresponds to the flag

$$V^\bullet : V^1 \subset \cdots \subset V^{n-1} \subset V^n \equiv \mathbb{C}^n,$$

where $V^i = \langle e_{n+1-i}, \dots, e_n \rangle$, and it comes from the longest permutation $e_i \to e_{n+1-i}$ (up to a sign).

In this way we are reduced to the non relative situation. Geometrically, a flag $W^\bullet$ in $Fl(\mathbb{C}^n)$ satisfies condition (2.2) if the fixed flag $V^\bullet$ sweeps a complete flag in each subspace W^j. On the other hand, it is just a computation to verify that in this case $W^\bullet$ is obtained from an element in $\mathrm{SL}_n(\mathbb{C})$ *all of whose principal minors are non singular,* that is an element in the big cell of $\mathrm{SL}_n(\mathbb{C})$. This element can always be factorized into a product of a lower unipotent matrix times an upper Borel one. It follows that the flag $W^\bullet$ is represented by that lower unipotent matrix, whose entries together with z are thus coordinates on X_0. Furthermore, let $p \in U_\alpha \cap U_\beta \subset C$ and let $W^\bullet \in X_0$ be a flag over p. It follows that $W^\bullet$ can be represented by a lower triangular matrix (frame) $\mathbf{N}_{-,\alpha}$ with respect to any trivialization and it is immediate that $\mathbf{N}_{-,\alpha}\varphi^0_{\alpha\beta}(p) = \varphi_{\alpha\beta}(p)\mathbf{N}_{-,\beta}$ in the overlap, where $\varphi^0_{\alpha\beta}$ is the diagonal part of $\varphi_{\alpha\beta}$. This relation shows that X_0 is an affine bundle over C.

2.3.2. It follows from (2.2) and the sketch of the proof above that $E|_{X_0}$ becomes (holomorphically) decomposable, hence we get at once the

Corollary 2.4.

$$\pi^* E|_{X_0} \cong \pi^* \widetilde{E}\Big|_{X_0} \qquad \square$$

Thus there are *frames* $\mathbf{M}$ in P at $p \in C$ — those projecting down to X_0 — that can be factorized into $\mathbf{M} = \mathbf{N}_-\mathbf{B}_+$, formally as in (1.9), and the entries of $\mathbf{N}_-$ are coordinates on $X_0 \cap X_p$. Moreover, $\mathbf{N}_-$ can be considered as a new frame in $\pi^* E$ which is "adapted" to the tautological flag $S^\bullet \hookrightarrow \pi^* E$.

3. Miura transformation and the horizontal foliation

3.1. In order to characterize the distribution $\mathcal{F}$ on X, recall that $\mathcal{F} \equiv \pi_* \mathcal{H}$, where $\mathcal{H}$ is the horizontal distribution in P determined by the connection ∇. Its "equations" are simply given by $\omega = 0$, where ω is the corresponding connection form. In particular we get $\omega^0 = \omega^+ = \omega^- = 0$, where ω^0, $\omega^\pm$ are the components of ω with respect to the root decomposition of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$.

Lemma 3.1. *The form ω^- is horizontal with respect to the fibration $P \xrightarrow{\tilde{\pi}} X$, hence it descends on X. The equation for $\mathcal{F}$ in X is $\omega^- = 0$.*

Proof. The relative flag X is gotten as a quotient of P by the action of B^+, so that we get $\omega^-(\xi)=0$ for any $\xi\in\mathfrak{b}^+$. (We identify elements of the Lie algebra with the corresponding fundamental vector fields.) Since they generate the vertical bundle of $P\xrightarrow{\tilde\pi}X$, the $\mathfrak{n}^-$–valued 1–form ω^- is *horizontal* and we can keep the same name for the corresponding object on X. It follows that an element of T_X is in $\mathcal{F}$ iff it annihilates ω^-. $\square$

3.2. We now introduce explicit coordinates and restrict our attention to $X_0\subset X$. If $p\in U\subset C$, with z a local parameter at p, we trivialize $P|_U\cong U\times \mathrm{SL}_n(\mathbb{C})$ with coordinates (z,g). Moreover, if $(z,g)\in P|_U\cap\tilde\pi^{-1}(X_0)$ then $g=n_-b_+$, with $b_+\in B^+$, $n_-\in N_-$. Thus $\tilde\pi: P|_U\cap\tilde\pi^{-1}(X_0)\to X_0|_U$ is locally given by $(z,g)\mapsto(z,n_-)$. Furthermore, let $\sigma_-:\mathbb{A}^{n(n-1)/2}\to N_-$ be the map sending an $n(n-1)/2$-tuple of complex numbers into a lower unipotent matrix. Since $(z,\underline{u})\in U\times\mathbb{A}^{n(n-1)/2}$ are local coordinates on $X_0|_U$, $(z,\underline{u})\mapsto(z,\sigma_-(\underline{u}))$ is a local section to $P\xrightarrow{\tilde\pi}X$. It follows that

$$\begin{gathered} U\times\mathbb{A}^{n(n-1)/2}\times B^+\longrightarrow P|_U\cap\tilde\pi^{-1}(X_0)\\ (z,\underline{u},b)\mapsto(z,\sigma_-(\underline{u})\,b)\end{gathered}$$

is a new choice of local coordinates on P. Putting

$$g=\sigma_-(\underline{u})\,b \tag{3.1}$$

is a coordinate change in the fibers of P over points in $X_0|_U\subset X$.

3.3. Let A be the local connection matrix for ∇ on U as given in (1.1), so that

$$\omega|_{P|_U}=g^{-1}dg+\mathrm{Ad}_{g^{-1}}(A) \tag{3.2}$$

is the local expression for the connection form, where $g^{-1}dg$ denotes the Maurer-Cartan form on $\mathrm{SL}_n(\mathbb{C})$. Using (3.1) we compute

$$\omega|_{P|_U}=b^{-1}db+\mathrm{Ad}_{b^{-1}}(\sigma_-^{-1}d\sigma_-+\mathrm{Ad}_{\sigma_-^{-1}}(A)) \tag{3.3}$$

and since $b\in B^+$ and $\sigma_-(\underline{u})\in N_-$ we have

$$\omega^-|_{P|_U}=\sigma_-^{-1}d\sigma_-+(\mathrm{Ad}_{\sigma_-^{-1}}(A))^- \tag{3.4}$$

where $(\cdot)^-$ denotes the projection from $\mathfrak{sl}_n(\mathbb{C})$ to $\mathfrak{n}^-$. Thus the differential ideal determining $\mathcal{F}$ on X_0 is generated by the entries of the $\mathfrak{n}^-$-valued 1-form $\omega^-|_{P|_U}$ (3.4) above.

The quantity in parentheses in (3.3) is interesting, insofar it is formally analogous to (1.3). Indeed, let us explicitly write $\nabla\mathbf{e}=\mathbf{e}\otimes A$ for ∇ on E and $\mathbf{e}=(e_1,\dots,e_n)$ the frame introduced in (2.1.2). Let us also keep the same notation for the corresponding object pulled back to $X|_U$ under π. From 2.3.2 and 3.2 we have that $\mathbf{e}\cdot\sigma_-(\underline{u})$ is a new frame for $\pi^*E\cong\pi^*\widetilde{E}$ on $X_0|_U$, and the new connection matrix

on $X_0|_U$ reads

$$\begin{aligned}\tilde{A} &= \sigma_-^{-1} d\sigma_- + \mathrm{Ad}_{\sigma_-^{-1}}(A) \\ &= \begin{pmatrix} u_1 & 1 & 0 & \dots & 0 \\ 0 & u_2 - u_1 & 1 & \dots & 0 \\ \dots & & \dots & & \dots \\ \dots\dots\dots\dots\dots\dots & & & u_{n-1} - u_{n-2} & 1 \\ 0 & \dots & \dots & 0 & -u_{n-1} \end{pmatrix} dz + \omega^-|_{P|_U} \end{aligned} \tag{3.5}$$

Now, $\tau : \Omega^1_X \to \mathcal{F}^*$ kills the annihilator of $\mathcal{F}$ and hence ω^-, therefore $\tau(\tilde{A})$ has the same form as the "connection matrix" in (1.2). It follows that the diagonal gauge operator is best interpreted as the partial connection operator δ on $X_0|_U$. Note that by construction the connection ∇ on $\pi^* E$ is adapted to δ, which is what we wanted to show.

4. Examples

4.1. Rank 2. The bundle E is the well known extension [12]

$$0 \longrightarrow K_C^{1/2} \longrightarrow E \longrightarrow K_C^{-1/2} \longrightarrow 0$$

with transition functions

$$\varphi_{\alpha\beta} = \begin{pmatrix} k_{\alpha\beta}^{-1/2} & 0 \\ \frac{d}{dz_\beta} k_{\alpha\beta}^{1/2} & k_{\alpha\beta}^{1/2} \end{pmatrix}$$

where $k_{\alpha\beta} = (dz_\alpha / dz_\beta)^{-1}$ are the transition functions for K_C. The connection ∇ has the form (see (1.1))

$$\nabla = \partial + \begin{pmatrix} 0 & 1 \\ T(z) & 0 \end{pmatrix} dz\,.$$

In this case $X = \mathbb{P}(E)$ and $X_0 = X \setminus C_\infty$, where C_∞ is the divisor at infinity, that is the image of C under the canonical section of the ruled surface X given by the sub-linebundle $K_C^{1/2}$. Points in X_0 are of the form $\left[\begin{smallmatrix} 1 \\ u \end{smallmatrix}\right]$ and the section $\sigma_- : X_0|_U \to P|_{X_0|U}$ is given by $\sigma_-(u) = \left(\begin{smallmatrix} 1 & 0 \\ u & 1 \end{smallmatrix}\right)$. It follows then that the 1-form defining the foliation on $X_0 \subset X$ has the expression

$$\omega^-|_{P|_U} = dw + (T(z) - w^2) dz \tag{4.1}$$

which should be compared with expression (1.6).

4.2. Rank 3. In this case $E = J^2(K_C^{-1})$. An explicit cocycle for it is [3]

$$\varphi_{\alpha\beta} = \begin{pmatrix} k_{\alpha\beta}^{-1} & 0 & 0 \\ 2\sigma_{\alpha\beta} & 1 & 0 \\ \partial^2 k_{\alpha\beta} & 2k_{\alpha\beta}\sigma_{\alpha\beta} & k_{\alpha\beta} \end{pmatrix}$$

where $k_{\alpha\beta}$ has the same meaning as before and $\sigma_{\alpha\beta} = \partial \log k_{\alpha\beta}^{1/2}$.

It follows from the general theory that X_0 consists of flags $W^1 \subset W^2$ such that $S^2_{W^\bullet} \cap (\pi^* F^1)_{W^\bullet} = \{0\}$ and $S^1_{W^\bullet} \cap \big(S^2_{W^\bullet} \cap (\pi^* F^2)_{W^\bullet}\big) = \{0\}$. Local coordinates are

(z, u_1, u_2, u_3) and $\sigma_- = \left(\begin{smallmatrix} 1 & 0 & 0 \\ u_1 & 1 & 0 \\ u_3 & u_2 & 1 \end{smallmatrix}\right)$ is the section of $P \xrightarrow{\tilde{\pi}} X$ on $X_0|_U$. The foliation is given by the following differential forms

$$\begin{aligned} \omega_1^- &= du_1 + (u_3 - u_1^2)\, dz \\ \omega_2^- &= du_2 + (T(z) - u_2^2 + u_1 u_2 - u_3)\, dz \\ \omega_3^- &= du_3 - u_2 du_1 + (w(z) + u_1 T(z) - u_2 u_3 + u_1^2 - u_1 u_3)\, dz \end{aligned}$$

and it is easily verified that the solution of this differential system is precisely (1.7).

5. Conclusions

We have pointed out how the global geometry underlying the Miura transformation is to be found in the foliation $\mathcal{F}$ in the flag bundle $X = Fl(E)$. As mentioned in the introduction, one of the problems concerning the definition of the Miura transformation on a Riemann surface was the monodromy behavior of its solutions. According to the point of view outlined here, this monodromy is nothing but the way the leaves are stacked in the foliation, thus it is an essential part of the overall structure. Since $\mathcal{F}$ comes from a flat structure, it is moreover just another way to look at the holonomy of the flat connection. The next step would be to set up a proper generalization of the free field quantization scheme in terms of this new interpretation of the Miura transformation. We hope to return to the subject elsewhere.

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